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Application of shehu transform to handling bessel function and cryptography

Mulugeta Andualem*, Atinafu Asfaw
Department of Mathematics, Bonga University, Bonga, Ethiopia

Abstract
Cryptography is the study of art and science of preparing protected and secure data communication. The word cryptography is derived from the two Greek words; "kryptos" means "secret or hidden" and "graphos" means "to write. In this study, we will discuss the Shehu transform method to solve Bessel's function of order p of first kind and encryption and decryption method.

Keywords: bessel function, encryption, cryptography, shehu transform

Introduction

Many problems in engineering and science can be formulated in terms of differential equations. The ordinary differential equations arise in many areas of Mathematics, as well as in Sciences and Engineering. In order to solve the certain ordinary differential equations integral transforms are widely used. In this paper, we will be discussed about the solution of Bessel's function of order $p$of first kind and encryption and decryption method using Shehu transform.

Shehu Transform

Definition

A new transform called the Shehu transform of the function $v(t)$ belonging to a class A, where:

$$A = \left\{ v(t): \exists\, N, \eta_1, \eta_2 > 0, |v(t)| < Ne^{\frac{|t|}{\eta_i}}, if\ t \in (-1)^i \times [0\ \infty) \right\}$$

Where $v(t)$ defined by $\mathbb{S}[v(t)]$ and is given by:

$$\mathbb{S}[v(t)] = V(s,u) = \int_0^\infty e^{\left(\frac{-st}{u}\right)} v(t)dt$$

And the inverse Shehu transform is defined as

$$\mathbb{S}^{-1}[V(s,u)] = v(t) \text{for } t \geq 0$$

Property of the Shehu Transform

1. Property 1. Linearity property of Shehu transform. Let the functions $\alpha v(t)$ and $\beta w(t)$ be in set A, then $(\alpha v(t) + \beta w(t)) \in A$, where α and β are nonzero arbitrary constants, and $\mathbb{S}[\alpha v(t) + \beta w(t)] = \alpha\mathbb{S}[v(t)] + \beta\mathbb{S}[w(t)]$

Proof: Using the Definition (1.1) of Shehu transform, we get

$$\mathbb{S}[\alpha v(t) + \beta w(t)] = \int_0^\infty e^{\left(\frac{-st}{u}\right)} (\alpha v(t) + \beta w(t))dt \quad 1.3$$

$$= \int_0^\infty e^{\left(\frac{-st}{u}\right)} \alpha v(t)dt \int_0^\infty e^{\left(\frac{-st}{u}\right)} \beta w(t)dt$$

$$= \alpha \int_0^\infty e^{\left(\frac{-st}{u}\right)} v(t)dt + \beta \int_0^\infty e^{\left(\frac{-st}{u}\right)} w(t)dt$$

$$= \alpha\mathbb{S}[v(t)] + \beta\mathbb{S}[w(t)]$$

Property 2. Let the function $v(\beta t)$ be in set A, where β is an arbitrary constant. Then

$$\mathbb{S}[\beta v(t)] = \frac{u}{\beta} V\left(\frac{s}{\beta}, u\right)$$

Using the Definition 1.1 of Shehu transform, we deduce

$$\mathbb{S}[\beta v(t)] = \int_0^\infty e^{\left(\frac{-st}{u}\right)} v(\beta t)dt \qquad 1.4$$

Substituting $x = \beta t \Rightarrow t = \frac{x}{\beta}$ and $\frac{dt}{dx} = \frac{1}{\beta} \Rightarrow dt = \frac{dx}{\beta}$ in equation 1.4 yield

$$\mathbb{S}[\beta v(t)] = \frac{1}{\beta}\int_0^\infty e^{\left(\frac{-sx}{u\beta}\right)} v(x)dx$$

$$= \frac{1}{\beta}\int_0^\infty e^{\left(\frac{-st}{u\beta}\right)} v(t)dt$$

$$= \frac{u}{\beta}\int_0^\infty e^{\left(\frac{-st}{\beta}\right)} v(ut)dt$$

$$V\left(\frac{s}{\beta}, u\right)$$

Derivative of Shehu transform. If the function $v^{(n)}(t)$ is the nth derivative of the function $v(t) \in A$ with respect to t, then its Shehu transform is defined by

$$\mathbb{S}\left[v^{(n)}(t)\right] = \frac{s^n}{u^n}V(s,u) - \sum_{k=0}^{n-1}\left(\frac{s}{u}\right)^{n-(k+1)} v^{(k)}(0) \qquad 1.5$$

When $n = 1$, we obtain the following derivatives with respect to t.

$$\mathbb{S}\left[v^{(1)}(t)\right] = \mathbb{S}[v'(t)] = \frac{s}{u}V(s,u) - v(0) \qquad 1.6$$

When $n = 2$, we obtain the following derivatives with respect to t.

$$\mathbb{S}\left[v^{(2)}(t)\right] = \mathbb{S}[v''(t)] = \frac{s^2}{u^2}V(s,u) - \frac{s}{u}v(0) - v'(0) \qquad 1.7$$

Assume that equation 1.5 true for $n = k$. Now we want to show that for $n = k + 1$

$$\mathbb{S}\left[v^{(k+1)}(t)\right] = \mathbb{S}[(v^{(k)}(t))'] = \frac{s}{u}S\left[v^{(k)}(t)\right] - v^{(k)}(0) \text{ using equation 1.6}$$

$$= \frac{s}{u}\left[\frac{s^k}{u^k}\mathbb{S}[v(t)] - \sum_{i=0}^{k-1}\left(\frac{s}{u}\right)^{k-(i+1)} v^{(i)}(0)\right] v^{(k)}(0)$$

$$= \left(\frac{s}{u}\right)^{k+1}\mathbb{S}[v(t)] - \sum_{i=0}^{k}\left(\frac{s}{u}\right)^{k-i} v^{(i)}(0)$$

Which implies that Eq (1.5) holds for
$n = k + 1$. By induction hypothesis the proof is complete

Property 3: Let the function $v(t) = 1$ be in set A. Then its Shehu transform is given by

$$\mathbb{S}[1] = \frac{u}{s}$$

Poof: Using equation 1.1

$$\mathbb{S}[1] = \int_0^\infty e^{\left(\frac{-st}{u}\right)} dt$$

$$= -\frac{u}{s}\lim_{\eta\to\infty}\left[e^{\left(\frac{-s\eta}{u}\right)}\right]_0^\infty = \frac{u}{s}$$

Property 4: Let the function $v(t) = sin(\alpha t)$ be in set A. Then its Shehu transform is given by

$$\mathbb{S}[sin(\alpha t)] = \frac{\alpha u^2}{s^2+\alpha^2u^2}$$

Property 5: Let the function $v(t) = cos(\alpha t)$ be in set A. Then its Shehu transform is given by

$\mathbb{S}[cos(\alpha t] = \frac{us}{s^2+\alpha^2u^2}$

Property 6: Let the function $v(t) = \ texp(\alpha t)$ and $v(t) = \ texp(\alpha t)$ be in set A. Then its Shehu transform is given by

$\frac{u^2}{(s-\alpha u)^2}$ and $\frac{u}{(s-\alpha u)}$ respectively

Property 7: Let the function $v(t) = t^n$ be in set A. Then its Shehu transform is given by

$\mathbb{S}[t^n\ ] = n!\left(\frac{u}{s}\right)^{n+1}$ for $n = 0, 1, 2 \ldots$

Shehu transform for handling Bessel functions

Bessel function is defined for a first time by the mathematician Daniel Bernoulli and generalized by Friedrich Bessel. A differential equation of the form

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - p^2)y = 0$$

Where p is arbitrary real or complex number is called a Bessel equation and its solution is known as Bessel function. Bessel's function of order p of first kind is defined as

$$J_p(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\left(\frac{t}{2}\right)^{p+2r}$$

For $p = 0, 1, 2$

$$J_0(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r)!}\left(\frac{t}{2}\right)^{2r} = 1 - \left(\frac{t}{2}\right)^2 + \frac{t^4}{2^24^2} - \frac{t^6}{2^24^26^2} + \cdots$$

$$J_1(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(1+r)!}\left(\frac{t}{2}\right)^{1+2r} = \frac{t}{2} - \frac{t^3}{2^24} + \frac{t^5}{2^24^2} - \frac{t^7}{2^24^26} + \cdots$$

$$J_2(t) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(2+r)!}\left(\frac{t}{2}\right)^{2+2r} = \frac{t^2}{22^2} - \frac{t^4}{2^22^26} + \frac{t^6}{2^24^262^3} - \frac{t^8}{2^24^26^22^3.10} + \cdots$$

Relationship between $J_0(t), J_1(t)$ and $J_2(t)$

$$\left(J_0(t)\right)' = \left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(r)!}\left(\frac{t}{2}\right)^{2r}\right)' = -\frac{t}{2} + \frac{t^3}{2^24} - \frac{t^5}{2^24^2} + \frac{t^7}{2^24^26} - \cdots$$

$$= -\left(\frac{t}{2} - \frac{t^3}{2^24} + \frac{t^5}{2^24^2} - \frac{t^7}{2^24^26} + \cdots\right)$$

$$= -J_1(t)$$

Therefore $(J_0(t))' = -J_1(t)$ and $J_2(t) = J_0(t) + 2J_0{}''(t)$

$$\mathbb{S}[J_p(t)] = \mathbb{S}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\left(\frac{t}{2}\right)^{p+2r}\right]$$

$$= \int_0^{\infty} e^{\left(\frac{-st}{u}\right)}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\left(\frac{t}{2}\right)^{p+2r}\right]dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\int_0^{\infty} e^{\left(\frac{-st}{u}\right)}\left(\frac{t}{2}\right)^{p+2r} dt$$

$$= \frac{1}{2^{p+2r}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\int_0^{\infty} e^{\left(\frac{-st}{u}\right)} t^{p+2r}dt$$

$$= \frac{1}{2^{p+2r}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}(p + 2r)!\left(\frac{u}{s}\right)^{(p+2r+1)}$$

Therefore, $\mathbb{S}[J_p(t)] = \mathbb{S}\left[\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}\left(\frac{t}{2}\right)^{p+2r}\right] = \frac{1}{2^{p+2r}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!(p+r)!}.(p + 2r)!\left(\frac{u}{s}\right)^{(p+2r+1)}$

Case 1: $p = 0$

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(r)!} \left(\frac{t}{2}\right)^{2r} = 1 - \left(\frac{t}{2}\right)^2 + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$$

Now take the Shehu transform both sides of the above result

$$\mathbb{S}[J_0(t)] = \mathbb{S}\left[1 - \left(\frac{t}{2}\right)^2 + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \right]$$

$$= \mathbb{S}[1] - \mathbb{S}\left[\left(\frac{t}{2}\right)^2\right] + \mathbb{S}\left[\frac{t^4}{2^2 4^2}\right] - \mathbb{S}\left[\frac{t^6}{2^2 4^2 6^2}\right] + \cdots$$

$$= \frac{u}{s} - \frac{1}{2^2} 2! \left(\frac{u}{s}\right)^3 + \frac{1}{2^2 4^2} 4! \left(\frac{u}{s}\right)^5 - \frac{1}{2^2 4^2 6^2} 6! \left(\frac{u}{s}\right)^7 + \cdots$$

$$= \frac{u}{s}\left[1 - \frac{1}{2}\left(\frac{u}{s}\right)^2 + \frac{1.3}{2.4}\left(\left(\frac{u}{s}\right)^2\right)^2 - \frac{1.3.3}{2.4.6}\left(\left(\frac{u}{s}\right)^3\right)^2 + \cdots\right]$$

$$= \frac{u}{s}\left(1 + \left(\frac{u}{s}\right)^2\right)^{\frac{-1}{2}}$$

$$\left[\frac{s}{\sqrt{u^2+s^2}}\right] = \frac{u}{\sqrt{u^2+s^2}}$$

Therefore, $\mathbb{S}[J_0(t)] = \frac{u}{\sqrt{u^2+s^2}}$

Case 2: $p = 1$

Since, $(J_0(t))' = -J_1(t)$. Now by applying the property of Shehu transform, we have

$$J_1(t) = -\left[\frac{s}{u}\mathbb{S}[J_0(t)] - J_0(0)\right]$$

$$= -\left[\frac{s}{u}\frac{u}{\sqrt{u^2+s^2}} - 1\right]$$

Since, $J_0(t) = 1 - \left(\frac{t}{2}\right)^2 + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$ implies $J_0(0) = 1$

$$\Rightarrow J_1(t) = -\left[\frac{s}{u}\frac{u}{\sqrt{u^2+s^2}} - 1\right]$$

$$J_1(t) = 1 - \frac{s}{\sqrt{u^2+s^2}}$$

Case 3: $p = 2$

Since, $J_2(t) = J_0(t) + 2J_0''(t)$ and $J_0(t) = 1 - \left(\frac{t}{2}\right)^2 + \frac{t^4}{2^2 4^2} - \frac{t^6}{2^2 4^2 6^2} + \cdots$ implies

$J_0'(t) = -t + \frac{t^3}{4^2} - \frac{t^5}{2^2 4^2 6} + \cdots$ and $J_0''(t) = -1 + \frac{3t^2}{4^2} - \frac{5t^4}{2^2 4^2 6} + \cdots$

Now by applying the property of Shehu transform of both sides of $J_2(t) = J_0(t) + 2J_0''(t)$, we have

$$\mathbb{S}[J_2(t)] = \mathbb{S}[J_0(t)] + 2\,\mathbb{S}[J_0''(t)]$$

$$= \frac{u}{\sqrt{u^2+s^2}} + 2\left[\frac{s^2}{u^2}J_0(t) - \frac{s}{u}J_0(0) - J_0'(0)\right]$$

$$= \frac{u}{\sqrt{u^2+s^2}} + 2\left[\frac{s^2}{u^2}\frac{u}{\sqrt{u^2+s^2}} - \frac{s}{u}\right]$$

$$= \frac{u}{\sqrt{u^2+s^2}} + \frac{2s^2}{u\sqrt{u^2+s^2}} - \frac{2s}{u}$$

$$= \frac{u^2+2s^2-2s\sqrt{u^2+s^2}}{u\sqrt{u^2+s^2}}$$

Shehu transform for handling Cryptography

In this section, we will disuse Shehu transform for encrypting the plain text and corresponding inverse Shehu transform is used for decryption.

Encryption Algorithm

A. Treat every letter in the plain text message as a number, so that $A = 1, B = 2, C = 3, \ldots Z = 26$, [space] = 0.

B. The plain text message is organized as finite sequence of numbers based on the above conversion. For example, our text is "BONGA". Based on the above step; we know that, B = 2, $O = 15$, N = 14, G = 7, A = 1 Therefore our plaintext finite sequence is 2, 15, 14, 7, 1

C. If $n + 1$ is the number of terms in the sequence; consider a polynomial of degree n with coefficient as the term of the given finite sequence. Above finite sequence contains $5 + 1$ terms. Hence consider a polynomial $p(t)$ of degree 4.

$$p(t) = 2 + 15t + 14t^2 + 7t^3 + t^4$$

Take the Shehu transform of the polynomial $p(t)$

$$\mathbb{S}[p(t)] = \mathbb{S}\left[2 + 15\frac{t}{1!} + 28\frac{t^2}{2!} + 42\frac{t^3}{3!} + 24\frac{t^4}{4!}\right]$$

$$[p(t)] = 2\mathbb{S}[1] + 15\mathbb{S}\left[\frac{t}{1!}\right] + 28\mathbb{S}\left[\frac{t^2}{2!}\right] + 42\mathbb{S}\left[\frac{t^3}{3!}\right] + 24\mathbb{S}\left[\frac{t^4}{4!}\right]$$

$$= 2\frac{u}{s} + 15\left(\frac{u}{s}\right)^2 + 28\left(\frac{u}{s}\right)^3 + 42\left(\frac{u}{s}\right)^4 + 24\left(\frac{u}{s}\right)^5$$

$$= \sum_{j=1}^{4+1} q_j \left(\frac{u}{s}\right)^{j+1}$$

Next find r_j such that $q_j \equiv r_j$ mod 26 for each j, $1 \le j \le n + 1$. Therefore

$q_1 = 2 \equiv 2$ mod 26, $q_2 = 15 \equiv 15$ mod 26, $q_3 = 28 \equiv 2$ mod 26, $q_4 = 42 \equiv 16$mod 26, $q_5 = 24 \equiv 24$ mod 26

D. Hence, $q_j = 26k_j + r_j$. Thus we get a key k_j for $j = 1, 2, \ldots, n + 1$

Therefore, $k_1 = 0, k_2 = 0, k_3 = 1, k_4 = 1, k_5 = 0$

E. Now consider a new finite sequence $r_1, r_1, \ldots, r_{n+1}$

That is, 2, 15, 2, 16, 24

Then the cipher text is "BOBPX"

Decryption Algorithm

1. Consider the cipher text and key received from sender. In the above example cipher text is "BOBPX" and key is 0, 0, 1, 1, 0
2. Convert the given cipher text to corresponding finite sequence of numbers $r_1,\ r_1,\ r_1, \ldots, r_{n+1}$
   2, 15, 2, 16, 24
3. Let $q_j = 26k_j + r_j, \forall j = 1, 2, \ldots, n + 1$
   $q_1 = 26(0) + 2 = 2, q_2 = 26(0) + 15 = 15,\ q_3 = 26(1) + 2 = 28,$
   $q_3 = 26(1) + 16 = 42, q_4 = 26(0) + 24 = 24$
4. $\sum_{j=0}^{4} q_j \left(\frac{u}{s}\right)^{j+1} = 2\left(\frac{u}{s}\right)^1 + 15\left(\frac{u}{s}\right)^2 + 28\left(\frac{u}{s}\right)^3 + 52\left(\frac{u}{s}\right)^4 + 24\left(\frac{u}{s}\right)^5$
5. Now take the Inverse Shehu transform of $2.0!\left(\frac{u}{s}\right)^1 + 15.1!\left(\frac{u}{s}\right)^2 + 14.2!\left(\frac{u}{s}\right)^3 + 7.3!\left(\frac{u}{s}\right)^4 + 1.4!\left(\frac{u}{s}\right)^5$ we obtain:

$$p(t) = 2 + 15t^1 + 14t^2 + 7t^3 + t^4$$

Consider the coefficient of a polynomial $p(t)$ as a finite sequence

$2, 15, 14, 7, 1$

Now translating the number of above finite sequence to alphabets. We get the original plain text as "BONGA"

Conclusion

In this paper, we have successfully discussed the Shehu transform of Bessel's functions and we have used Shehu transform for encrypting the plain text and corresponding inverse Shehu transform for decryption.

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